MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Dr. S. Bachmann / Dr. P. Soneji

WS 2014/15
– Blatt 6 –
19.11.2014

PDG I
(Zentralübung)

Problem Sheet 6

Question 1

Prove Theorem 34 from the lectures: consider the half-space

$$\mathbb{R}^n_+ := \{(x_1, \dots, x_n \in \mathbb{R}^n : x_n > 0\} .$$

Suppose $f \in C^2_c(\mathbb{R}^n_+)$ and $g \in C^0(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. As in the lectures, define the Green function for $\mathbb{R}^n_+$ by

$$G(x, y) := \Phi(x - y) - \Phi(\tilde{x} - y)$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the fundamental solution to the Laplace equation and

$$\tilde{x} := (x_1, \dots, x_{n-1}, -x_n) , \quad x \in \mathbb{R}^n_+ .$$

Now define

$$v(x) := \begin{cases} \int_{\mathbb{R}^n_+} f(y)\, G(x, y)\, \mathrm{d}y - \int_{\partial\mathbb{R}^n_+} g(y)\, \frac{\partial G}{\partial \nu}(x, y)\, \mathrm{d}S(y) & x \in \mathbb{R}^n_+ \\ g(x) & x \in \partial\mathbb{R}^n_+ . \end{cases}$$

Then

(i) $v \in C^2(\mathbb{R}^n_+)$.

(ii) v satisfies

$$\begin{cases} -\Delta v(x) = f(x) & x \in \mathbb{R}^n_+ \\ v(x) = g(x) & x \in \partial\mathbb{R}^n_+ . \end{cases}$$

(iii) We have

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} v(x) = g(x_0)$$

for all $x_0 \in \partial\mathbb{R}^n_+$.

Question 2

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with C^1 boundary. For a function $w \in C^1(\overline{\Omega})$ the n-dimensional surface area over its graph

$$\{(x, w(x)) : x \in \overline{\Omega}\}$$

is given by the functional

$$A(w) := \int_\Omega \sqrt{1 + |Dw(x)|^2}\, dx\,.$$

Let $g \in C(\partial\Omega)$ and suppose that $u \in C^2(\overline{\Omega})$ is a minimiser of A within the set

$$\{w \in C^1(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}\,.$$

Prove that this minimiser u solves the boundary-value problem

$$\begin{cases} \operatorname{div}\left(\frac{Du(x)}{\sqrt{1+|Du(x)|^2}}\right) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega\,, \end{cases}$$

which is called the *minimal surface equation.*

Deadline for handing in: 0800 Wednesday 26 November

Please put solutions in Box 17, 1st floor (near the library)

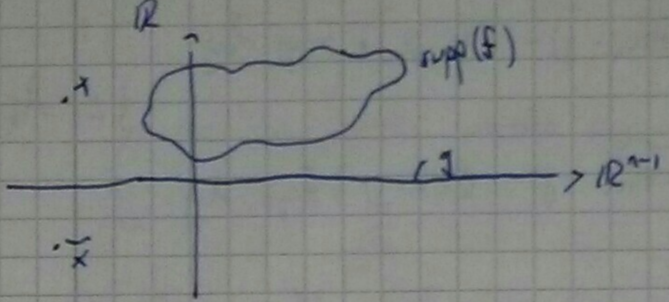
Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`

Sheet 6

① Satz 34 : $f \in C^2_c(\mathbb{R}^n_+)$, $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$

$G(x,y) := \Phi(x-y) - \Phi(\tilde{x}-y)$ $\left[x \mapsto \Phi(\tilde{x}-y) =: \varphi^y(x) \quad (y \in \mathbb{R}^n_+ \text{ fixed}) \right.$ solves $\left. \begin{cases} \varphi^y(x) = \Phi(x-y) & \text{on } \partial\mathbb{R}^n_+ \\ \Delta \varphi^y(x) = 0 & \text{on } \mathbb{R}^n_+ \end{cases} \right]$

$\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$

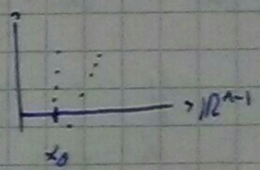


let $v(x) := \begin{cases} \int_{\mathbb{R}^n_+} f(y)\, G(x,y)\, dy - \int_{\partial\mathbb{R}^n_+} \frac{\partial G}{\partial \nu}(x,y)\, g(y)\, dS(y) \\ g(x) \end{cases}$

Show: (i) $v \in C^2(\mathbb{R}^n_+)$

(ii) $\begin{cases} -\Delta v(x) = f(x) & x \in \mathbb{R}^n_+ \\ v(x) = g(x) & x \in \partial\mathbb{R}^n_+ \quad (\text{by def}) \end{cases}$

(iii) $\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} v(x) = g(x) \quad \forall x_0 \in \partial\mathbb{R}^n_+$



Note $G(x,y) = G(y,x)$ $\quad x,y \in \mathbb{R}^n_+$

Also $\frac{\partial G}{\partial y_i}(x,y) = \frac{-1}{\omega_n}\left[\frac{y_i - x_i}{|y_i - x_i|^n} - \frac{y_i - \tilde{x}_i}{|y_i - \tilde{x}_i|^n} \right]$ $\begin{cases} \tilde{x}_i = x_i & \text{if } 1 \le i \le n-1 \\ \tilde{x}_n = -x_n \end{cases}$

For $y \in \partial\mathbb{R}^n_+$: $= \begin{cases} 0 & \text{if } y \in \partial\mathbb{R}^n_+, \ 1 \le i \le n-1 \\ \frac{1}{\omega_n}\frac{2x_n}{|y-x|^n} & i = n \end{cases}$

$\nu(y) = (0, \dots, 0, -1)$ $\qquad \downarrow \nu(y)$

Hence $\frac{\partial G}{\partial \nu}(x,y) = -\frac{2x_n}{\omega_n} \frac{1}{|x-y|^n} = -K(x,y)$ (from Sheet 5, Q 2(c) (Poisson's kernel for $\mathbb{R}^n_+$.))

So $\quad v(x) = \int_{\mathbb{R}^n_+} f(y)\, G(x,y)\, dy + \int_{\partial \mathbb{R}^n_+} g(y)\, K(x,y)\, dS(y)$

(ii) Show $v \in C^2(\mathbb{R}^n_+)$:

Let $1 \le i \le n$ and $h \in \mathbb{R}$; fix $x \in \mathbb{R}^n_+$. Then

$$\frac{v(x+he_i) - v(x)}{h} = \left.\begin{aligned} &\frac{1}{h}\int_{\mathbb{R}^n_+} [\Phi(x+he_i-y) - \Phi(x-y)]\, f(y)\, dy \\ &- \frac{1}{h}\int_{\mathbb{R}^n_+} [\Phi(\widetilde{x+he_i}-y) - \Phi(\tilde{x}-y)]\, f(y)\, dy \end{aligned}\right\} =: A_h$$

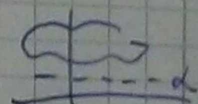
$$+ \left.\frac{1}{h}\int_{\partial\mathbb{R}^n_+} [K(x+he_i,y) - K(x,y)]\, g(y)\, dS(y)\right\} =: B_h$$

$$A_h = \frac{1}{h}\int_{\mathbb{R}^n_+} [\Phi(x+he_i) - \Phi(x)]\, f(y)\, dy - \frac{1}{h}\int_{\mathbb{R}^n_+} [\Phi(x+he_i-\tilde{y}) - \Phi(x-\tilde{y})]\, f(y)\, dy$$

symmetry of G.

$\operatorname{supp} f \subset\subset \mathbb{R}^n_+$. So $\exists\, \alpha > 0$ s.t. $\operatorname{supp} f \subset\subset \{z \in \mathbb{R}^n_+ \mid z_n > \alpha\}$

Hence, for $|h| < \alpha$, ~~taking~~ using a change of variables,



$$A_h = \cancel{\frac{1}{h}} \int_{\mathbb{R}^n_+} \underbrace{[\Phi(x-y) - \Phi(\tilde{x}-y)]}_{\varphi(y) \in L^1(\mathbb{R}^n_+)} \underbrace{\frac{f(y+he_i) - f(y)}{h}}_{\psi_h(y)}\, dy$$

$\psi_h(y) \to \frac{\partial f}{\partial y_i}(y) \quad \forall y \in \mathbb{R}^n_+$ uniformly in $\mathbb{R}^n_+$. $(f \in C^2_c)$

Hence, using DCT,

$$A_h \to \int_{\mathbb{R}^n_+} G(x,y) \frac{\partial f}{\partial y_i}(y)\, dy.$$

Now consider B_h: for fixed $x \in \mathbb{R}^n_+$, $y \in \partial\mathbb{R}^n_+$,

$$\chi_h(y) := \frac{K(x+he_i,y) - K(x,y)}{h}\, g(y) \to g(y) \frac{\partial}{\partial x_i} K(x,y) \quad \text{as } h \to 0.$$

Note that $|y-x| \geq x_n$

Here $|K(x,y)| \leq \dfrac{C}{x_n^{n-1}}$

Similarly $\left|\dfrac{\partial}{\partial x_i} K(x,y)\right| \leq \begin{cases} \dfrac{C}{x_n^{n+1}} & 1 \leq i \leq n-1 \\ \dfrac{C}{x_n^{n+2}} \cdot \dfrac{C}{x_n^{n-1}} & i=n. \end{cases}$

K and its derivatives bounded by a constant depending on x_n. $|K_h(y)| \leq C\|g\|_\infty$

So we can apply DCT and have

$$B_h \to \int_{\partial \mathbb{R}^n_+} \frac{\partial}{\partial x_i} K(x,y)\, g(y)\, dS(y)$$

So $\dfrac{\partial v}{\partial x_i}$ exists. Arguing in the same way, we can also show

$\dfrac{\partial v}{\partial x_i x_j}$ exists and equals

$$\int_{\mathbb{R}^n_+} G(x,y) \frac{\partial^2 f}{\partial y_i \partial y_j}(y)\, dy + \int_{\partial \mathbb{R}^n_+} \frac{\partial^2}{\partial x_i \partial x_j} K(x,y)\, g(y)\, dS(y).$$

So $v \in C^2(\mathbb{R}^n_+)$.

(ii) Show $-\Delta v(x) = f(x)$, $x \in \mathbb{R}^n_+$.

By (i), fixing $x \in \mathbb{R}^n_+$.

$$\Delta v(x) = \int_{\mathbb{R}^n_+} G(x,y)\, \Delta_y f(y)\, dy + \int_{\partial \mathbb{R}^n_+} \Delta_x K(x,y)\, g(y)\, dy$$

$$\Delta_x K(x,y) = 0 \quad \text{(straightforward)}$$

Now take $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subset\subset \mathbb{R}^n_+$.

Then $\Delta v(x) = \underbrace{\int_{\mathbb{R}^n_+ \setminus B(x,\varepsilon)} G(x,y)\,\Delta_y f(y)\,dy}_{I_\varepsilon} + \underbrace{\int_{B(x,\varepsilon)} G(x,y)\,\Delta_y f(y)\,dy}_{J_\varepsilon}$

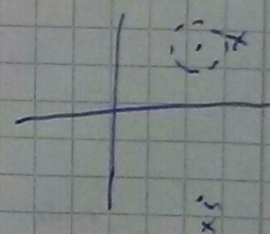
$|J_\varepsilon| \le C\|D^2 f\|_\infty \int_{B(x,\varepsilon)} |G(x,y)|\,dy$

$\left(n \max_{\substack{z\in\mathbb{R}^n_+ \\ 1\le i,j\le n}} |f_{y_i y_j}(z)| \right)$

$\le C\|D^2 f\|_\infty \int_{B(x,\varepsilon)} |\Phi(x-y)| + |\Phi(\tilde{x}-y)|\,dy$

$\int_{B(x,\varepsilon)} |\Phi(x-y)|\,dy \le \begin{cases} C\,\varepsilon^{2}\log\varepsilon & n=2 \\ C\,\varepsilon^{n}/\varepsilon^{n-2} = C\varepsilon^2 & n\ge 3 \end{cases} \longrightarrow 0$ as $\varepsilon\to 0$.

$\int_{B(x,\varepsilon)} |\Phi(\tilde{x}-y)|\,dy \le \begin{cases} C\varepsilon^2\log(2x_n+\varepsilon) & n=2 \\ C\varepsilon^n / |2x_n-\varepsilon|^{n-2} \end{cases}$



$\longrightarrow 0$ as $\varepsilon \to 0$.

So $|J_\varepsilon| \to 0$ as $\varepsilon \to 0$.

$I_\varepsilon = \int_{\mathbb{R}^n_+ \setminus B(x,\varepsilon)} G(x,y)\,\Delta_y f(y)\,dy$

Gauss-Green / int by parts / c.f. Sheet 2

$= \underbrace{\int_{\partial(\mathbb{R}^n_+ \setminus B(x,\varepsilon))} G(x,y)\frac{\partial f}{\partial \nu}(y)\,dy}_{K_\varepsilon} - \underbrace{\int_{\mathbb{R}^n_+ \setminus B(x,\varepsilon)} \nabla_y G(x,y)\cdot\nabla f(y)\,dy}_{L_\varepsilon}$

$\partial(\mathbb{R}^n_+ \setminus B(x,\varepsilon)) = \partial\mathbb{R}^n_+ \cup \partial B(x,\varepsilon)$

For $y\in\partial\mathbb{R}^n_+$, $G(x,y) = \Phi(x-y) - \Phi(\tilde{x}-y) = 0$ $\quad (|x-y| = |\tilde{x}-y|)$

So

$$|K_\varepsilon| = \left| \int_{\partial B(x,\varepsilon)} G(x,y) \frac{\partial f}{\partial \nu}(y)\, d_y S(y) \right|$$

$$\leq C \|Df\|_\infty \int_{\partial B(x,\varepsilon)} |G(x,y)|\, dS(y)$$

$$\int_{\partial B(x,\varepsilon)} |G(x,y)|\, dS(y) \leq \int_{\partial B(x,\varepsilon)} |\Phi(x-y)|\, dS + \int_{\partial B(x,\varepsilon)} |\Phi(\tilde{x}-y)|\, dS(y)$$

$$\int_{\partial B(x,\varepsilon)} |\Phi(x-y)|\, dS(y) \leq \begin{cases} C\varepsilon \log \varepsilon & n=2 \\ C\varepsilon & n\geq 3 \end{cases} \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

$$\int_{\partial B(x,\varepsilon)} \Phi(\tilde{x}-y)\, |dS(y)| \leq \begin{cases} C\varepsilon \log(2x_n+\varepsilon) & n=2 \\ \dfrac{C\varepsilon^{n-1}}{|2x_n-\varepsilon|^{n-2}} & n\geq 3 \end{cases} \longrightarrow 0 \text{ as } \varepsilon \to 0$$

So $|K_\varepsilon| \to 0$ as $\varepsilon \to 0$.

$$L_\varepsilon = \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \Delta_y G(x,y) f(y)\, dy - \int_{\partial B(x,\varepsilon)} f(y) \underbrace{\frac{\partial G(x,y)}{\partial \nu}}_{= \nabla_y(\Phi(x-y)}\, dS(y)$$

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y \left(\Phi(x-y) - \Phi(\tilde{x}-y)\right) \cdot \nu(y)$$

inward unit $\nu = -\dfrac{(y-x)}{|y-x|}$

$$= \frac{1}{\omega_n} \frac{(y-x)\cdot(y-x)}{|y-x|^n\,|y-x|} = \frac{1}{\omega_n} \varepsilon^{-(n-1)} \quad \text{for } y \in \partial B(x,\varepsilon)$$

Hence $L_\varepsilon = -\int_{\partial B(x,\varepsilon)} f(y) \dfrac{1}{\omega_n \varepsilon^{n-1}} = -⨍_{\partial B(x,\varepsilon)} f(y)\, dS(y)$

$$\longrightarrow -f(x) \quad \text{as } \varepsilon \to 0.$$

Hence, taking $\varepsilon \to 0$ in (*),

$$\Delta u(x) = -f(x)$$

(iii) Show $\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} v(x) = g(x_0) \quad \forall x_0 \in \partial \mathbb{R}^n_+$.

Let $x_0 \in \partial \mathbb{R}^n_+$

Note $\int_{\mathbb{R}^n_+} G(x,y) f(y)\, dy = \int_{\mathbb{R}^n_+} \big(\underbrace{\Phi(x-y) - \Phi(\tilde{x}-y)} \big) f(y)\, dy$

$\longrightarrow \Phi(x_0 - y) - \Phi(x_0 - y) = 0$ as $x \to x_0$

Using $f \in C^2_c$, DCT,

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} \int_{\mathbb{R}^n_+} G(x,y) f(y)\, dy = 0.$$

Remains to show $\int_{\partial \mathbb{R}^n_+} K(x,y) g(y)\, dS(y) \longrightarrow g(x_0)$.

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $|g(y) - g(x_0)| < \varepsilon$ if $y \in \partial \mathbb{R}^n_+$, $|y - x_0| < \delta$.

Then if $|x - x_0| < \frac{\delta}{2}$ $(x \in \mathbb{R}^n_+)$,

Sheet 5 Q2(c): $\int_{\partial \mathbb{R}^n_+} K(x,y)\, dS(y) = 1$!

$$|u(x) - g(x_0)| = \left| \int_{\partial \mathbb{R}^n_+} K(x,y) \big(g(y) - g(x_0) \big)\, dS(y) \right|$$

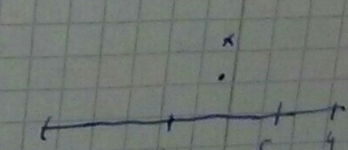
$$\le \int_{\partial \mathbb{R}^n_+ \cap B(x_0,\delta)} K(x,y) \underbrace{|g(y) - g(x_0)|}_{< \varepsilon}\, dS(y) + \int_{\partial \mathbb{R}^n_+ \setminus B(x_0,\delta)} K(x,y) |g(y) - g(x_0)|\, dS(y)$$

$< I + J$, say

$$I \le \varepsilon \int_{\partial \mathbb{R}^n_+} K(x,y)\, dS(y) = \varepsilon$$

Now note that if $|x - x_0| < \frac{\delta}{2}$, $|y - x_0| \ge \delta$, then

$$|y - x_0| \le |y - x| + \frac{\delta}{2} \le |y - x| + \frac{1}{2}|y - x_0|$$



So $|y - x| \ge \frac{1}{2}|y - x_0|$

Thus $J \le 2\|g\|_\infty \int_{\partial \mathbb{R}^n_+ \setminus B(x_0,\delta)} K(x,y)\, dS(y)$

$$\frac{2 x_n}{|y-x|^n} \le \frac{2^n x_n}{|y-x_0|^n}$$

$$\leq \frac{2^{n+2} x_n \|g\|_\infty}{\omega_n} \int_{\partial \mathbb{R}^n_+ \setminus B(x_0,\delta)} |y - x_0|^{-n}$$

$\longrightarrow 0$ as $x_n \downarrow 0$. (ie $< \varepsilon$ if $|x - x_0| < \tilde{\delta}$)

Hence, $|u(x) - g(x_0)| < 2\varepsilon$ provided $|x - x_0|$ small enough. $\square$

(2) $A(w) = \int_\Omega (1 + |\nabla w(x)|^2)^{\frac{1}{2}} \, dx$

$(1 + g(t))^{\frac{1}{2}}$

$\frac{g'(t)}{(1+g(t))^{\frac{1}{2}}}$

u minimises A $\quad u = g$ on $\partial\Omega$. $\qquad \varphi \in C^2_c(\Omega)$

$h(t) = A(u + t\varphi)$

$(1 + |\nabla u + t\varphi|^2)^{\frac{1}{2}}$

$(1 + (\sum_{i=1}^n (u_{x_i} + t\varphi_{x_i})^2))^{\frac{1}{2}}$

$= (1 + (\sum_{i=1}^n u_{x_i}^2 + 2t u_{x_i}\varphi_{x_i} + t^2 \varphi_{x_i}^2))^{\frac{1}{2}}$

$\frac{d}{dt} = \frac{\sum \frac{1}{2}(\cancel{2} u_{x_i}\varphi_{x_i} + \cancel{2} t \varphi_{x_i}^2)}{(1 + |\nabla(u + t\varphi)|^2)^{\frac{1}{2}}}$

$= \frac{\nabla u \cdot \nabla\varphi + t|\nabla\varphi|^2}{(1 + |\nabla(u+t\varphi)|^2)^{\frac{1}{2}}}$

$h'(0) = \int_\Omega \frac{\nabla u \cdot \nabla \varphi}{(1 + |\nabla u|^2)^{\frac{1}{2}}} \, dx = 0.$

Gauss-Green: $\int_\Omega \operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right) \varphi \, dx = 0$

$\varphi = 0$ on $\partial\Omega$

Prob Sheet 7:

Evans p87 • Q. 13. Derive heat kernel in 1d by scaling

Evans p87 • Q.14 (Ex 5, tutorial sheet 9) - tutorial problem ⟵ (b)

• Sheet print out Qu: do for $x \in \mathbb{R}^n$. (a)

Hint. recall methods used for transport equation. 2

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u = 0 & \cancel{\mathbb{R}^n \times (0,\infty)}\ (0,\infty) \times \mathbb{R}^n \\ u(\cancel{t}\, 0, x) = g(x) & x \in \mathbb{R}^n \end{cases}$$

$g \in C^\infty(\cancel{\mathbb{R}}) \cap L^\infty(\mathbb{R}^n)$ $\qquad$ $b \in \mathbb{R}^n$ fixed

Write down formula for solution for this.
(involving heat kernel)

Tutorial: $\boxed{\text{Mollifiers}}$. Heat kernel is a mollifier!
(it's the standard

cf Sheet ~~3~~ 2 qu 3

$\varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right)$

If $\varphi \in C_c^\infty(B)$ mollifier

f cont $\cancel{\Delta}$: $\cancel{\varphi}$ $(\varphi_\varepsilon * f)(x) \longrightarrow f(x)$.

} something about derivative / density.

—

1 Evans Q 13

2 (a) Print out
(b) Evans Q14 (Ex 5, tutorial sheet 9)
} Hint. Recall TE.

* Note: ~~this related to~~ Black Scholes Model!
The following equation are
n=1 the ~~follows the~~ is closely related to

$n=1$ — (c) $u_t - \Delta u + b \cdot \nabla u + cu = 0$ Black Scholes Model
$\quad\quad$ b and c
u - option price x - stock price, t time.